

# CONVOLUTIONS OF SLANTED HALF-PLANE HARMONIC MAPPINGS

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**ABSTRACT.** Let  $\mathcal{S}^0(H_\gamma)$  denote the class of all univalent, harmonic, sense-preserving and normalized mappings  $f$  of the unit disk  $\mathbb{D}$  onto the slanted half-plane  $H_\gamma := \{w : \operatorname{Re}(e^{i\gamma}w) > -1/2\}$  with an additional condition  $f_{\overline{z}}(0) = 0$ . Functions in this class can be constructed by the shear construction due to Clunie and Sheil-Small which allows by examining their conformal counterpart. Unlike the conformal case, convolution of two univalent harmonic convex mappings in  $\mathbb{D}$  is not necessarily even univalent in  $\mathbb{D}$ . In this paper, we fix  $f_0 \in \mathcal{S}^0(H_0)$  and show that the convolutions of  $f_0$  and some slanted half-plane harmonic mapping are still convex in a particular direction. The results of the paper enhance the interest among harmonic mappings and, in particular, solves an open problem of Dorff, et. al. [5] in a more general setting. Finally, we present some basic examples of functions and their corresponding convolution functions with specified dilatations, and illustrate them graphically with the help of MATHEMATICA software. These examples explain the behaviour of the image domains.

## 1. INTRODUCTION

In this paper, we consider the class  $\mathcal{H}$  of complex-valued harmonic functions  $f = h + \overline{g}$  defined on the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , where  $h$  and  $g$  are analytic on  $\mathbb{D}$  with the form

$$(1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

If we write  $f = u + iv$ , then  $u$  and  $v$  are real harmonic in  $\mathbb{D}$ . Moreover, the Jacobian of  $f = h + \overline{g}$  is given by  $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ . Lewy's theorem implies that every harmonic function  $f$  on  $\mathbb{D}$  is locally one-to-one and sense-preserving on  $\mathbb{D}$  if and only if  $J_f(z) > 0$  in  $\mathbb{D}$ . The condition  $J_f(z) > 0$  is equivalent to the existence of an analytic function  $\omega$  in  $\mathbb{D}$  such that

$$(2) \quad |\omega(z)| < 1 \quad \text{for } z \in \mathbb{D},$$

where  $\omega(z) = g'(z)/h'(z)$  which is referred to as the *complex dilatation* of  $f$ . By requiring harmonic function to be sense-preserving we retain some basic properties exhibited by

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analytic functions, such as the open mapping property, the argument principle, and zeros being isolated (see [7]).

During the last two decades, after the publication of landmark paper of Clunie and Sheil-Small [2], the class  $\mathcal{S}_H$  of sense-preserving univalent functions  $f \in \mathcal{H}$  together with its subclasses have been extensively studied. Let  $\mathcal{S}_H^0$  be the subset of all  $f \in \mathcal{S}_H$  in which  $b_1 = f_{\bar{z}}(0) = 0$ . We remark that the familiar class  $\mathcal{S}$  of normalized analytic univalent functions is contained in  $\mathcal{S}_H^0$ . Every  $f \in \mathcal{S}_H$  admits the complex dilatation  $\omega$  of  $f$  which satisfies (2). When  $f \in \mathcal{S}_H^0$ , we also have  $\omega'(0) = 0$ . Finally, let  $\mathcal{K}_H^0$ ,  $\mathcal{S}_H^{*0}$ , and  $\mathcal{C}_H^0$  denote the subclasses of  $\mathcal{S}_H^0$  mapping  $\mathbb{D}$  onto, respectively, convex, starlike, and close-to-convex domains, just as  $\mathcal{K}$ ,  $\mathcal{S}^*$ , and  $\mathcal{C}$  are the subclasses of  $\mathcal{S}$  mapping  $\mathbb{D}$  onto these respective domains. The reader is referred to [2, 6] for many interesting results and expositions on planar univalent harmonic mappings.

Although not much is known in the literature on results about harmonic convolution of functions, some progress has been achieved in the recent years, see [4, 5, 8]. We define the harmonic convolution (or Hadamard product) as follows: For  $f = h + \bar{g} \in \mathcal{H}$  with the series expansions for  $h$  and  $g$  as above in (1), and  $F = H + \bar{G} \in \mathcal{H}$ , where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B_n z^n,$$

we define

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}.$$

Clearly,  $f * F = F * f$ . In the case of conformal mappings, the literature about convolution theory is exhaustive. For example, we have [14]

$$\mathcal{K} * \mathcal{K} \subset \mathcal{K}, \quad \mathcal{S}^* * \mathcal{K} \subset \mathcal{S}^*, \quad \mathcal{C} * \mathcal{K} \subset \mathcal{C}$$

settling the Pólya-Schoenberg conjecture. For some related containment relations, we refer to [11, 12] and many other later works of Ruscheweyh. Unfortunately, these inclusion results do not necessarily carryover to harmonic mappings. In fact, in view of the sharp coefficient bounds for functions in  $\mathcal{K}_H^0$ , if we take  $f$  and  $F$  in the class  $\mathcal{K}_H^{*0}$ , then it will not always be true that  $F * f \in \mathcal{K}_H^{*0}$  (does not necessary be even univalent). On the other hand, based on the question raised by Clunie and Sheil-Small [2], several authors have studied the subclass of functions  $f \in \mathcal{S}_H^0$  that map  $\mathbb{D}$  onto specific domains such as horizontal strips, see [9]. A function  $f = h + \bar{g} \in \mathcal{S}_H^0$  is called a slanted half-plane mapping with  $\gamma$  ( $0 \leq \gamma < 2\pi$ ) if  $f$  maps  $\mathbb{D}$  onto  $H_\gamma := \{w : \operatorname{Re}(e^{i\gamma} w) > -1/2\}$ . Using the shearing method due to Clunie and Sheil-Small [2], it is almost easy to obtain that such a mapping has the form (see [5, Lemma 1])

$$(3) \quad h(z) + e^{-2i\gamma} g(z) = \frac{z}{1 - e^{i\gamma} z}.$$

We denote by  $\mathcal{S}^0(H_\gamma)$ , the class of all slanted half-plane mappings with  $\gamma$ . In the harmonic case, one can easily see that there are infinitely many slanted half-plane mapping with a fixed  $\gamma$ .

For  $\gamma = 0$ , we get the class of right half-plane mappings  $f$  that map  $\mathbb{D}$  onto  $f(\mathbb{D}) = H_0 = \{w : \operatorname{Re} w > -1/2\}$  and such mappings clearly assume the form

$$h(z) + g(z) = \frac{z}{1-z}.$$

Moreover if  $f \in \mathcal{K}_H^0$  and  $\phi \in \mathcal{S}^0(H_0)$ , then  $f * \phi$  is not necessarily belong to  $\mathcal{S}_H^0$ . This can be easily seen to be true even if  $\phi \in \mathcal{S}^0(H_\gamma)$  for any  $\gamma$ .

Throughout the paper  $f_0 = h_0 + \overline{g_0}$ , where

$$h_0(z) = \frac{z - z^2/2}{(1-z)^2} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n = \frac{1}{2} \left( \frac{z}{1-z} + \frac{z}{(1-z)^2} \right)$$

and

$$g_0(z) = \frac{-z^2/2}{(1-z)^2} = -\sum_{n=2}^{\infty} \frac{n-1}{2} z^n = \frac{1}{2} \left( \frac{z}{1-z} - \frac{z}{(1-z)^2} \right).$$

The function  $f_0$ , which acts as extremal for many issues concerning the convex class  $\mathcal{K}_H^0$ , has the dilatation  $\omega(z) = -z$ . Moreover, if  $f = h + \overline{g} \in \mathcal{H}$ , then the above representation for  $h_0$  and  $g_0$  quickly gives that

$$f_0 * f = h_0 * h + \overline{g_0 * g} = \frac{h + zh'}{2} + \frac{\overline{g - zg'}}{2}.$$

This fact will be used in the proof of Lemma 1 while determining the dilatation of the convolution functions. Clearly  $f_0 \in \mathcal{S}^0(H_0)$ , because  $h_0(z) + g_0(z) = \frac{z}{1-z}$  and  $f_0 \in \mathcal{K}_H^0$  (see [2]). We observe that  $f_0 * f_0 \notin \mathcal{K}_H^0$ , see [2, Theorem 5.7].

A domain  $\Omega \subset \mathbb{C}$  is said to be *convex in the direction*  $\gamma$ ,  $\gamma \in \mathbb{R}$ , if and only if for every  $a \in \mathbb{C}$ , the set  $\Omega \cap \{a + te^{i\gamma} : t \in \mathbb{R}\}$  is either connected or empty. Dorff et. al. [5] proved

**Theorem A.** ([5, Theorem 2]) *If  $f_k \in \mathcal{S}^0(H_{\gamma_k})$ ,  $k = 1, 2$ , and  $f_1 * f_2$  is locally univalent in  $\mathbb{D}$ , then  $f_1 * f_2$  is convex in the direction  $-(\gamma_1 + \gamma_2)$ .*

Theorem A generalizes the result of Dorff [4, Theorem 5] who proved it when  $\gamma_1 = \gamma_2 = 0$ . Moreover, Theorem A implies that for every  $f \in \mathcal{S}^0(H_\gamma)$ ,  $f * f_0$  is convex in the direction  $-\gamma$  provided that convolution function is locally univalent in  $\mathbb{D}$ . On the other hand, the following result deals with two cases for which the local univalence of the resulting convolution function as demanded in Theorem A is not necessary.

**Theorem B.** ([5, Theorem 3]) *Let  $f = h + \overline{g} \in \mathcal{S}^0(H_0)$  with the dilatation  $\omega(z) = e^{i\theta} z^n$  ( $n = 1, 2$ ),  $\theta \in \mathbb{R}$ . Then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction of the real axis.*

We now state our first result which shows that Theorem B continues to hold in the general setting.

**Theorem 1.** *Let  $f = h + \overline{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = e^{i\theta} z^n$ , where  $n = 1, 2$  and  $\theta \in \mathbb{R}$ . Then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction  $-\gamma$ .*

Also, we present an example when the local univalence fails for  $\omega(z) = e^{i\theta} z^n$  if  $n \geq 3$ . We note that if  $f \in \mathcal{S}^0(H_\gamma)$  with  $\gamma = \pi$ , then we get the class of left half-plane mappings

$f$  that map  $\mathbb{D}$  onto  $f(\mathbb{D}) = H_\pi = \{w : \operatorname{Re} w < 1/2\}$  so that

$$h(z) + g(z) = \frac{z}{1+z}.$$

Setting  $\gamma = \pi$  in Theorem 1 gives

**Corollary 1.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\pi)$  with the dilatation  $\omega(z) = e^{i\theta} z^n$ , where  $n = 1, 2$  and  $\theta \in \mathbb{R}$ . Then  $f_0 * f \in \mathcal{S}_H^0$ .*

Recently, Bshouty and Lyzzaik [1] brought out a collection of open problems and conjectures on planar harmonic mappings, proposed by many colleagues throughout the past quarter of a century. In [1, Problem 3.26(a)], Dorff et. al. posed the following open question.

**Problem.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_0)$  with the dilatation  $\omega(z) = (z + a)/(1 + \bar{a}z)$ ,  $(-1 < a < 1)$ . Then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction of the real axis ([5, Theorem 4]). Determine other values of  $a \in \mathbb{D}$  for which the previous result holds.*

In [10], the present authors have solved this problem. In continuation of our investigation, in this paper we consider this problem in a general setting by allowing  $f$  to vary in  $\mathcal{S}^0(H_\gamma)$ .

**Theorem 2.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = \frac{z+a}{1+\bar{a}z}$ , where  $a = |a|e^{i\theta}$ ,  $\theta = \arg z$  and  $|a| < 1$ . If*

$$|a|^2 \left( \cos^2 \left( \theta - \frac{\gamma}{2} \right) + 9 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) \leq 1$$

*but the condition*

$$|a| \cos \left( \theta - \frac{\gamma}{2} \right) = -\cos \frac{3\gamma}{2} \quad \text{and} \quad 3|a| \sin \left( \theta - \frac{\gamma}{2} \right) = -\sin \frac{3\gamma}{2}$$

*does not hold, then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction  $-\gamma$ .*

Theorems 1 and 2 supplement the works of Dorff et. al. [5] and, Clunie and Sheil-Small [2]. Finally, in Section 3 we include important special cases of Theorem 2 which includes a solution to the Problem of Dorff et. al. [1] (see Corollary 4).

## 2. MAIN LEMMAS

**Lemma 1.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = g'(z)/h'(z)$ . Then the dilatation  $\tilde{\omega}$  of  $f_0 * f$  is*

$$(4) \quad \tilde{\omega}(z) = -ze^{-i\gamma} \left( \frac{\omega^2(z) + e^{2i\gamma}[\omega(z) - \frac{1}{2}z\omega'(z)] + \frac{1}{2}e^{i\gamma}\omega'(z)}{1 + e^{-2i\gamma}[\omega(z) - \frac{1}{2}z\omega'(z)] + \frac{1}{2}e^{-i\gamma}z^2\omega'(z)} \right).$$

*Proof.* Assume the hypothesis that  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with  $\omega(z) = g'(z)/h'(z)$ . Then

$$g'(z) = \omega(z)h'(z) \quad \text{and} \quad g''(z) = \omega'(z)h'(z) + \omega(z)h''(z).$$

Moreover, as  $f$  satisfies the condition (3), the first equality above gives

$$(5) \quad h'(z) = \frac{1}{(1 + e^{-2i\gamma}\omega(z))(1 - e^{i\gamma}z)^2}$$

and therefore,

$$(6) \quad h''(z) = \frac{-(1 - e^{i\gamma}z)e^{-2i\gamma}\omega'(z) + 2(1 + e^{-2i\gamma}\omega(z))e^{i\gamma}}{(1 + e^{-2i\gamma}\omega(z))^2(1 - e^{i\gamma}z)^3}.$$

From the representation of  $h_0$  and  $g_0$ , we see that

$$(h_0 * h)(z) = \frac{h(z) + zh'(z)}{2} \quad \text{and} \quad (g_0 * g)(z) = \frac{g(z) - zg'(z)}{2}.$$

Therefore, as  $f_0 * f = h_0 * h + \overline{g_0 * g}$ , we have

$$(7) \quad \tilde{\omega}(z) = \frac{(g_0 * g)'(z)}{(h_0 * h)'(z)} = -\frac{zg''(z)}{2h'(z) + zh''(z)} = -\frac{z\omega'(z)h'(z) + \omega(z)zh''(z)}{2h'(z) + zh''(z)}.$$

In view of (5) and (6), after some computation (7) takes the desired form.  $\square$

The case  $\gamma = 0$  of Lemma 1 apparently used in the proof of Theorem 3 in [5] whereas the case  $\gamma = \pi$  of Lemma 1 gives

**Corollary 2.** *Let  $f = h + \overline{g} \in \mathcal{S}^0(H_\pi)$  with the dilatation  $\omega(z)$ . Then the dilatation  $\tilde{\omega}$  of  $f_0 * f$  is given by*

$$\tilde{\omega}(z) = z \frac{\omega^2(z) + [\omega(z) - \frac{1}{2}z\omega'(z)] - \frac{1}{2}\omega'(z)}{1 + [\omega(z) - \frac{1}{2}z\omega'(z)] - \frac{1}{2}z^2\omega'(z)}.$$

The following lemma is required for the proof of Theorem 1.

**Lemma 2.** *Let  $f = h + \overline{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = g'(z)/h'(z) = e^{i\theta}z^n$  ( $n = 1, 2$  and  $\theta \in \mathbb{R}$ ). Then the dilatation of  $f_0 * f$  is*

$$(8) \quad \tilde{\omega}(z) = -z^n e^{(2\theta-\gamma)i} \left( \frac{z^{n+1} + e^{(2\gamma-\theta)i}(1 - \frac{n}{2})z + \frac{n}{2}e^{(\gamma-\theta)i}}{1 + e^{(\theta-2\gamma)i}(1 - \frac{n}{2})z^n + \frac{n}{2}e^{(\theta-\gamma)i}z^{n+1}} \right).$$

*Proof.* Consider  $\omega(z) = e^{i\theta}z^n$ . Then  $\omega'(z) = ne^{i\theta}z^{n-1}$ . Using these, (4) gives

$$\tilde{\omega}(z) = -ze^{-i\gamma} \left( \frac{e^{2i\theta}z^{2n} + e^{2i\gamma}[e^{i\theta}z^n - \frac{1}{2}zne^{i\theta}z^{n-1}] + \frac{1}{2}e^{i\gamma}ne^{i\theta}z^{n-1}}{1 + e^{-2i\gamma}[e^{i\theta}z^n - \frac{1}{2}zne^{i\theta}z^{n-1}] + \frac{1}{2}e^{-i\gamma}z^2ne^{i\theta}z^{n-1}} \right)$$

and a simplification gives the desired formula (8).  $\square$

**Example 1.** The range of the dilatation function  $\tilde{\omega}$  in Lemma 2 is not contained in the unit disk  $\mathbb{D}$  if we assume  $n \geq 3$ . To see this, we choose  $\omega(z) = -z^n$ . Then (8) reduces to

$$\tilde{\omega}(z) = -z^n e^{-i\gamma} \left( \frac{z^{n+1} + e^{2i\gamma}(\frac{n}{2} - 1)z - \frac{n}{2}e^{i\gamma}}{1 + e^{-2i\gamma}(\frac{n}{2} - 1)z^n - \frac{n}{2}e^{-i\gamma}z^{n+1}} \right) = -z^n e^{-i\gamma} R(z) \quad (\text{say}).$$

It is a simple exercise to see that

$$|R(e^{i\alpha})| = 1 \quad \text{and} \quad R(z)(\overline{R(1/\overline{z})}) = 1$$

so that the function  $R(z)$  maps the closed disk  $|z| \leq 1$  onto itself and hence,  $R$  can be written as a finite Blaschke product of order  $n + 1$ . On the other hand, the product of the moduli of the zeros of  $R$  in the unit disk  $\mathbb{D}$  is  $n/2$ . This means that there exists a point  $z_0 \in \mathbb{D}$  such that  $|\tilde{\omega}(z_0)| > 1$  if  $n \geq 3$ . Thus, the restriction on  $n$ , namely,  $n = 1, 2$  in Lemma 2 becomes necessary for our investigations.

**Lemma 3.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = \frac{z+a}{1+\bar{a}z}$ , where  $|a| < 1$ . Then the dilatation  $\tilde{\omega}$  of  $f_0 * f$  is*

$$\tilde{\omega}(z) = -ze^{-i(\gamma-\phi)} \cdot \frac{(z+A)(z+B)}{(1+\bar{A}z)(1+\bar{B}z)},$$

where  $\phi = \arg((1 + \bar{a}e^{2i\gamma})/(1 + ae^{-2i\gamma}))$ , and

$$(9) \quad t(z) = z^2 + \frac{4a + e^{2i\gamma}(1 + 3|a|^2)}{2(1 + \bar{a}e^{2i\gamma})}z + \frac{2a^2 + 2ae^{2i\gamma} + e^{i\gamma}(1 - |a|^2)}{2(1 + \bar{a}e^{2i\gamma})}.$$

Here  $-A, -B$  are the two roots of  $t(z) = 0$ , and  $A, B$  may be equal. (The dilatation is well-defined provided  $|A|, |B| \leq 1$  which will be discussed in the next lemma)

*Proof.* We have

$$\omega'(z) = \frac{1 - |a|^2}{(1 + \bar{a}z)^2} \quad \text{for } |a| < 1.$$

In view of  $\omega(z) = \frac{z+a}{1+\bar{a}z}$  and the last equation, by (4), the dilatation  $\tilde{\omega}$  of  $f_0 * f$  takes the form

$$\tilde{\omega}(z) = -ze^{-i\gamma}W(z),$$

where

$$\begin{aligned} W(z) &= \frac{\frac{(z+a)^2}{(1+\bar{a}z)^2} + e^{2i\gamma}[\frac{z+a}{1+\bar{a}z} - \frac{1}{2}z\frac{1-|a|^2}{(1+\bar{a}z)^2}] + \frac{1}{2}e^{i\gamma}\frac{1-|a|^2}{(1+\bar{a}z)^2}}{1 + e^{-2i\gamma}[\frac{z+a}{1+\bar{a}z} - \frac{1}{2}z\frac{1-|a|^2}{(1+\bar{a}z)^2}] + \frac{1}{2}e^{-i\gamma}z^2\frac{1-|a|^2}{(1+\bar{a}z)^2}} \\ &= \frac{2(z+a)^2 + e^{2i\gamma}[2(z+a)(1+\bar{a}z) - z(1-|a|^2)] + e^{i\gamma}(1-|a|^2)}{2(1+\bar{a}z)^2 + e^{-2i\gamma}[2(z+a)(1+\bar{a}z) - z(1-|a|^2)] + e^{-i\gamma}(1-|a|^2)z^2} \\ &= \frac{2(1+\bar{a}e^{2i\gamma})z^2 + [4a + e^{2i\gamma}(1+3|a|^2)]z + 2a^2 + 2ae^{2i\gamma} + e^{i\gamma}(1-|a|^2)}{2(1+ae^{-2i\gamma}) + [4\bar{a} + e^{-2i\gamma}(1+3|a|^2)]z + [2\bar{a}^2 + 2\bar{a}e^{-2i\gamma} + e^{-i\gamma}(1-|a|^2)]z^2} \\ &= \left( \frac{1 + \bar{a}e^{2i\gamma}}{1 + ae^{-2i\gamma}} \right) \left( \frac{z^2 + \frac{4a+e^{2i\gamma}(1+3|a|^2)}{2(1+\bar{a}e^{2i\gamma})}z + \frac{2a^2+2ae^{2i\gamma}+e^{i\gamma}(1-|a|^2)}{2(1+\bar{a}e^{2i\gamma})}}{1 + \frac{4\bar{a}+e^{-2i\gamma}(1+3|a|^2)}{2(1+ae^{-2i\gamma})}z + \frac{2\bar{a}^2+2\bar{a}e^{-2i\gamma}+e^{-i\gamma}(1-|a|^2)}{2(1+ae^{-2i\gamma})}z^2} \right) \\ &= \left( \frac{1 + \bar{a}e^{2i\gamma}}{1 + ae^{-2i\gamma}} \right) \frac{t(z)}{t^*(z)}. \end{aligned}$$

Here  $t(z)$  is given by (9) and

$$t^*(z) = z^2 \overline{t(1/\bar{z})} = 1 + \frac{4\bar{a} + e^{-2i\gamma}(1 + 3|a|^2)}{2(1 + ae^{-2i\gamma})}z + \frac{2\bar{a}^2 + 2\bar{a}e^{-2i\gamma} + e^{-i\gamma}(1 - |a|^2)}{2(1 + ae^{-2i\gamma})}z^2.$$

Suppose that  $-A, -B$  are the two roots of  $t(z) = 0$  ( $A, B$  may be equal). Then

$$t(z) = (z+A)(z+B)$$

and

$$t^*(z) = z^2 \overline{t(1/\bar{z})} = z^2 \cdot \overline{(1/\bar{z} + A)(1/\bar{z} + B)} = (1 + \bar{A}z)(1 + \bar{B}z).$$

As  $|(1 + \bar{a}e^{2i\gamma})/(1 + ae^{-2i\gamma})| = 1$ , the desired form for  $\tilde{\omega}(z)$  follows.  $\square$

**Lemma 4.** Let  $t(z)$  be defined by (9) so that  $t(z) = (z + A)(z + B)$ . Also, let  $a = |a|e^{i\theta}$ , where  $\theta = \arg a$  with  $|a| < 1$ . If

$$(10) \quad |a|^2 \left( \cos^2 \left( \theta - \frac{\gamma}{2} \right) + 9 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) \leq 1,$$

then  $|AB| \leq 1$ . Moreover,  $|AB| = 1$  if and only if

$$(11) \quad |a| \cos \left( \theta - \frac{\gamma}{2} \right) = -\cos \frac{3\gamma}{2} \quad \text{and} \quad 3|a| \sin \left( \theta - \frac{\gamma}{2} \right) = -\sin \frac{3\gamma}{2}.$$

*Proof.* By the definition of  $t(z) = (z + A)(z + B)$ , it is clear that

$$AB = \frac{2a^2 + 2ae^{2i\gamma} + e^{i\gamma}(1 - |a|^2)}{2(1 + \bar{a}e^{2i\gamma})} = \frac{2a(a + e^{2i\gamma}) + e^{i\gamma}(1 - |a|^2)}{2(1 + \bar{a}e^{2i\gamma})}.$$

We look for a condition on  $a \in \mathbb{D}$  such that  $|AB| \leq 1$ . Now, a computation leads to

$$|2a(a + e^{2i\gamma}) + e^{i\gamma}(1 - |a|^2)|^2 - 4|1 + \bar{a}e^{2i\gamma}|^2 = (1 - |a|^2)v(a),$$

where  $v(a)$  is real and

$$v(a) = 4\operatorname{Re}(a^2e^{-i\gamma}) + 4\operatorname{Re}(a(e^{i\gamma} - 2e^{-2i\gamma})) - 3 - 5|a|^2.$$

Now, we let  $a = |a|e^{i\theta}$ . Then,  $v(a)$  reduces to

$$(12) \quad v(a) = 4|a|^2 \cos(2\theta - \gamma) + 4|a| \cos(\theta + \gamma) - 8|a| \cos(\theta - 2\gamma) - 3 - 5|a|^2.$$

If we use the cosine doubling formula  $\cos 2\phi = 1 - 2\sin^2 \phi$ , and then replace  $\theta + \gamma$  and  $\theta - 2\gamma$  respectively by  $\theta - \frac{\gamma}{2} + \frac{3\gamma}{2}$  and  $\theta - \gamma - \frac{3\gamma}{2}$ , by a simplification,  $v(a)$  takes the form

$$v(a) = - \left( |a| \cos \left( \theta - \frac{\gamma}{2} \right) + 2 \cos \frac{3\gamma}{2} \right)^2 - \left( 3|a| \sin \left( \theta - \frac{\gamma}{2} \right) + 2 \sin \frac{3\gamma}{2} \right)^2 + 1.$$

By (10), we observe that  $P_1(|a| \cos(\theta - \frac{\gamma}{2}), 3|a| \sin(\theta - \frac{\gamma}{2}))$  is a point that lies on the closed disk  $|z| \leq 1$  whereas the point  $P_2(-2 \cos \frac{3\gamma}{2}, -2 \sin \frac{3\gamma}{2})$  lies on the circle  $|z| = 2$ . Thus, the distance between the points  $P_1$  and  $P_2$  must be at least 1. That is,

$$\sqrt{\left( |a| \cos \left( \theta - \frac{\gamma}{2} \right) + 2 \cos \frac{3\gamma}{2} \right)^2 + \left( 3|a| \sin \left( \theta - \frac{\gamma}{2} \right) + 2 \sin \frac{3\gamma}{2} \right)^2} \geq 1$$

which is equivalent to saying that  $v(a) \leq 0$ , i.e  $|AB| \leq 1$ . Moreover, in the above inequality, equality holds if and only if the point  $P_1$  is the middle point of the line segment joining  $P_2$  and the origin. This gives the condition (11). In other words, if (10) holds but not the (11), then  $v(a) < 0$  and hence, strict inequality  $|AB| < 1$  holds. If (11) holds, then  $v(a) = 0$  and hence,  $|AB| = 1$ . The proof is complete.  $\square$

### 3. PROOFS OF MAIN THEOREMS AND THEIR CONSEQUENCES

**Proof of Theorem 1.** In view of Theorem A, it suffices to show that  $f_1 * f_2$  is locally univalent in  $\mathbb{D}$ . To prove this, first we consider the case  $n = 1$  so that  $\omega(z) = e^{i\theta}z$ . Then, by the formula (8), the dilatation  $\tilde{\omega}$  of  $f_0 * f$  becomes

$$\tilde{\omega}(z) = -ze^{(2\theta-\gamma)i} \left( \frac{z^2 + \frac{1}{2}e^{(2\gamma-\theta)i}z + \frac{1}{2}e^{(\gamma-\theta)i}}{1 + \frac{1}{2}e^{(\theta-2\gamma)i}z + \frac{1}{2}e^{(\theta-\gamma)i}z^2} \right) = -ze^{(2\theta-\gamma)i} \frac{t(z)}{t^*(z)},$$

where  $t(z) = z^2 + \frac{1}{2}e^{(2\gamma-\theta)i}z + \frac{1}{2}e^{(\gamma-\theta)i}$  and

$$t^*(z) = z^2 \overline{t(1/\bar{z})} = 1 + \frac{1}{2}e^{(\theta-2\gamma)i}z + \frac{1}{2}e^{(\theta-\gamma)i}z^2.$$

Clearly if  $z_0$  is a zero of  $t(z)$ , then  $1/\bar{z}_0$  is a zero of  $t^*(z)$ . Therefore, we may write the last expression as

$$\tilde{\omega}(z) = -ze^{(2\theta-\gamma)i} \frac{(z+A)(z+B)}{(1+\bar{A}z)(1+\bar{B}z)}.$$

We observe that  $A$  and  $B$  are nonzero complex numbers such that

$$A+B = \frac{1}{2}e^{(2\gamma-\theta)i} \quad \text{and} \quad AB = \frac{1}{2}e^{(\gamma-\theta)i}.$$

It is easy to see that  $A, B \in \overline{\mathbb{D}}$ . Observe that  $|\frac{1}{2}e^{(\gamma-\theta)i}| < 1$  and the only zero of

$$\frac{t(z) - \frac{1}{2}e^{(\gamma-\theta)i}t^*(z)}{z} = \frac{3}{4}z + \frac{1}{2}e^{(2\gamma-\theta)i} - \frac{1}{4}e^{-i\gamma},$$

namely,  $\frac{1}{3}e^{-i\gamma} - \frac{2}{3}e^{(2\gamma-\theta)i}$ , clearly lies in  $\overline{\mathbb{D}}$ . According to Cohn's Rule ([3] or see [13]), the two zeros of  $t(z)$ , namely  $-A$  and  $-B$ , must lie in  $\overline{\mathbb{D}}$ . This observation gives that  $|\tilde{\omega}(z)| < 1$  in  $\mathbb{D}$ .

Next, we consider the case  $n = 2$  so that  $\omega(z) = e^{i\theta}z^2$ . In this case, the formula (8) takes the form

$$\tilde{\omega}(z) = -z^2e^{(2\theta-\gamma)i} \left( \frac{z^3 + e^{(\gamma-\theta)i}}{1 + e^{-(\gamma-\theta)i}z^3} \right),$$

which clearly implies that  $|\tilde{\omega}(z)| < 1$  for  $z \in \mathbb{D}$ . According to Lewy's theorem, it turns out that  $f_1 * f_2$  is locally univalent in  $\mathbb{D}$  and hence, the desired conclusion follows from Theorem A.  $\square$

**Proof of Theorem 2.** By Lemma 4 and the hypothesis, we have  $|AB| < 1$ . Then at least one of  $A, B$  is in  $\mathbb{D}$ . Without loss of generality, we may assume that  $A \in \mathbb{D}$ . Next, consider the function  $t(z)$  defined by (9) in the form

$$t(z) = z^2 + a_1z + a_0 = (z+A)(z+B).$$

Then the function

$$t_1(z) = \frac{t(z) - a_0t^*(z)}{z} = (1 - |a_0|^2)z + a_1 - a_0\bar{a}_1$$

has a zero at

$$(13) \quad z_0 = \frac{a_0\bar{a}_1 - a_1}{1 - |a_0|^2} = \frac{A(|B|^2 - 1) + B(|A|^2 - 1)}{1 - |AB|^2},$$



which, after simplification, is equivalent to

$$(14) \quad z_0 = e^{-i\gamma} \frac{6a^2 e^{-i\gamma} + 8ae^{i\gamma} - 4\bar{a}e^{2i\gamma} - 3|a|^2 + 2e^{3i\gamma} - 1}{4\operatorname{Re}(a^2 e^{-i\gamma}) + 4\operatorname{Re}(a(e^{i\gamma} - 2e^{-2i\gamma})) - 3 - 5|a|^2}.$$

**Claim (a).**  $B \in \overline{\mathbb{D}}$  if and only if  $|z_0| \leq 1$ .

By a routine computation, it can be easily seen that

$$|A(|B|^2 - 1) + B(|A|^2 - 1)|^2 - (1 - |AB|^2)^2 = -(1 - |A|^2)(1 - |B|^2)|1 - A\bar{B}|^2.$$

As  $|AB| < 1$  and  $|A| < 1$ , the last equation and (13) show that Claim (a) holds. Indeed  $|B| < 1$  if and only if  $|z_0| < 1$ , and  $|B| = 1$  if and only if  $|z_0| = 1$ .

**Claim (b).**  $|z_0| \leq 1$  if and only if (10) holds.

We may conveniently write  $z_0$  as

$$z_0 = e^{-i\gamma} \frac{u(a)}{v(a)}$$

where  $v(a)$  is defined by (12) and

$$u(a) = 6a^2 e^{-i\gamma} + 8ae^{i\gamma} - 4\bar{a}e^{2i\gamma} - 3|a|^2 + 2e^{3i\gamma} - 1.$$

We observe that  $|z_0| \leq 1$  if and only if  $|u(a)| \leq |v(a)|$ , i.e.  $|u(a)|^2 - |v(a)|^2 \leq 0$ . In order to deal with the later inequality, we consider

$$(15) \quad |u(a)|^2 - |v(a)|^2 = (\operatorname{Im} u(a))^2 + (\operatorname{Re} u(a) - v(a))(\operatorname{Re} u(a) + v(a))$$

and each term needs to be simplified. First we find that

$$\operatorname{Re} u(a) = 6|a|^2 \cos(2\theta - \gamma) + 8|a| \cos(\theta + \gamma) - 4|a| \cos(\theta - 2\gamma) - 3|a|^2 + 2 \cos 3\gamma - 1,$$

and

$$\begin{aligned} \operatorname{Im} u(a) &= 6|a|^2 \sin(2\theta - \gamma) + 8|a| \sin(\theta + \gamma) + 4|a| \sin(\theta - 2\gamma) + 2 \sin 3\gamma \\ &= 12|a|^2 \sin\left(\theta - \frac{\gamma}{2}\right) \cos\left(\theta - \frac{\gamma}{2}\right) + 12|a| \sin\left(\theta - \frac{\gamma}{2}\right) \cos \frac{3\gamma}{2} \\ &\quad + 4|a| \cos\left(\theta - \frac{\gamma}{2}\right) \sin \frac{3\gamma}{2} + 4 \sin \frac{3\gamma}{2} \cos \frac{3\gamma}{2} \\ &= 4 \left( |a| \cos\left(\theta - \frac{\gamma}{2}\right) + \cos \frac{3\gamma}{2} \right) \left( 3|a| \sin\left(\theta - \frac{\gamma}{2}\right) + \sin \frac{3\gamma}{2} \right). \end{aligned}$$

Also, we see that

$$\begin{aligned} \operatorname{Re} u(a) - v(a) &= 2|a|^2 (\cos(2\theta - \gamma) + 1) + 4|a| [\cos(\theta + \gamma) + \cos(\theta - 2\gamma)] + 2(\cos 3\gamma + 1) \\ &= 4|a|^2 \cos^2\left(\theta - \frac{\gamma}{2}\right) + 8|a| \cos\left(\theta - \frac{\gamma}{2}\right) \cos \frac{3\gamma}{2} + 4 \cos^2 \frac{3\gamma}{2} \\ &= 4 \left( |a| \cos\left(\theta - \frac{\gamma}{2}\right) + \cos \frac{3\gamma}{2} \right)^2 \end{aligned}$$

and similarly,

$$\begin{aligned} \operatorname{Re} u(a) + v(a) &= 2|a|^2 (5 \cos(2\theta - \gamma) - 4) + 12|a| [\cos(\theta + \gamma) - \cos(\theta - 2\gamma)] + 2(\cos 3\gamma - 2) \\ &= 2 \left[ |a|^2 \left( 1 - 10 \sin^2\left(\theta - \frac{\gamma}{2}\right) \right) - 12|a| \sin\left(\theta - \frac{\gamma}{2}\right) \sin \frac{3\gamma}{2} - 2 \sin^2 \frac{3\gamma}{2} - 1 \right]. \end{aligned}$$

Using the above expressions, (15) takes the form

$$\begin{aligned}
|u(a)|^2 - |v(a)|^2 &= 8 \left[ |a| \cos \left( \theta - \frac{\gamma}{2} \right) + \cos \frac{3\gamma}{2} \right]^2 \left[ 2 \left( 3|a| \sin \left( \theta - \frac{\gamma}{2} \right) + \sin \frac{3\gamma}{2} \right)^2 \right. \\
&\quad \left. + |a|^2 \left( 1 - 10 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) - 12|a| \sin \left( \theta - \frac{\gamma}{2} \right) \sin \frac{3\gamma}{2} - 2 \sin^2 \frac{3\gamma}{2} - 1 \right] \\
&= 8 \left[ |a| \cos \left( \theta - \frac{\gamma}{2} \right) + \cos \frac{3\gamma}{2} \right]^2 \left[ \left( \cos^2 \left( \theta - \frac{\gamma}{2} \right) + 9 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) |a|^2 - 1 \right].
\end{aligned}$$

Since  $|a| < 1$ , the last equality shows that  $|z_0| < 1$  if and only if

$$|a|^2 \left( \cos^2 \left( \theta - \frac{\gamma}{2} \right) + 9 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) < 1.$$

Also,  $|z_0| = 1$  if and only if

$$|a|^2 \left( \cos^2 \left( \theta - \frac{\gamma}{2} \right) + 9 \sin^2 \left( \theta - \frac{\gamma}{2} \right) \right) = 1.$$

In conclusion, the assumption and Claims **(a)** and **(b)** imply that  $A \in \mathbb{D}$  and  $B \in \overline{\mathbb{D}}$ . We obtain that  $|\tilde{\omega}(z)| < 1$  for each  $z \in \mathbb{D}$ . Thus, by Theorem A, we deduce that  $f_0 * f \in \mathcal{S}_H^0$  and  $f_0 * f$  is convex in the direction  $-\gamma$ .  $\square$

**Corollary 3.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with the dilatation  $\omega(z) = \frac{z+a}{1+\bar{a}z}$ , where  $a = |a|e^{i\theta}$ ,  $\theta = \arg z$ ,  $|a| < 1$  and  $\gamma \in \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$ . If (10) holds, then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction  $-\gamma$ .*

*Proof.* If  $\gamma \in \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$  in Lemma 4, then obviously

$$\left| \cos \frac{3\gamma}{2} \right| = 1$$

and so the first equality in (11) holds if and only if  $|a| = 1$ , which contradicts the fact that  $|a| < 1$ . In other words, if  $\gamma \in \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$  then  $|AB| \neq 1$  and so in this case, the strict inequality  $|AB| < 1$  holds under the condition (10). This completes the proof.  $\square$

The case  $\gamma = 0$  of Corollary 3 gives the following result (see [10, Theorem]).

**Corollary 4.** *Let  $f = h + \bar{g} \in \mathcal{S}^0(H_0)$  with the dilatation  $\omega(z) = \frac{z+a}{1+\bar{a}z}$  ( $|a| < 1$ ). If*

$$(\operatorname{Re} a)^2 + 9(\operatorname{Im} a)^2 \leq 1,$$

*then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction of the real axis.*

Finally, the case  $\gamma = \pi$  of Theorem 2 gives

**Corollary 5.** *Let  $f = h + \bar{g} \in \mathcal{S}_H^0$  with  $h(z) + g(z) = \frac{z}{1+z}$  and the dilatation  $\omega(z) = \frac{z+a}{1+\bar{a}z}$ , where  $|a| < 1$  with  $\operatorname{Im} a \neq 0$ . If*

$$9(\operatorname{Re} a)^2 + (\operatorname{Im} a)^2 \leq 1,$$

*then  $f_0 * f \in \mathcal{S}_H^0$  and is convex in the direction of the real axis.*

## 4. SOME EXAMPLES

**Example 2.** Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with  $\gamma = \pi/2$  and the dilatation  $\omega(z) = z$ . Then

$$h(z) - g(z) = \frac{z}{1-iz} \quad \text{and} \quad g'(z) = zh'(z).$$

Solving these yield

$$h'(z) = \frac{1}{(1-z)(1-iz)^2} = \frac{1-i}{2} \cdot \frac{1}{(1-iz)^2} + \frac{1}{2} \left( \frac{1}{1-iz} + \frac{i}{1-z} \right).$$

Integration from 0 to  $z$  gives

$$h(z) = \frac{i}{2} \log \left( \frac{1-iz}{1-z} \right) + \frac{1-i}{2} \frac{z}{1-iz}$$

and therefore,

$$g(z) = \frac{i}{2} \log \left( \frac{1-iz}{1-z} \right) - \frac{1+i}{2} \frac{z}{1-iz}.$$

By the convolution, we have

$$f_0 * f = h_0 * h + \overline{g_0 * g}$$

so that

$$h_0 * h = \frac{h + zh'}{2} = \frac{i}{4} \log \frac{1-iz}{1-z} + \frac{1-i}{4} \frac{z}{1-iz} + \frac{z}{2(1-z)(1-iz)^2}$$

and

$$g_0 * g = \frac{g - zg'}{2} = \frac{i}{4} \log \frac{1-iz}{1-z} - \frac{1+i}{4} \frac{z}{1-iz} - \frac{z^2}{2(1-z)(1-iz)^2}.$$

Applying Lemma 1 with  $\gamma = \pi/2$ , we get

$$\tilde{\omega}(z) = iz \left( \frac{z^2 - \frac{1}{2}z + \frac{i}{2}}{1 - \frac{1}{2}z - \frac{i}{2}z^2} \right).$$

The real and imaginary parts of  $f_0 * f = h_0 * h + \overline{g_0 * g}$  may be written explicitly as follows:

$$\operatorname{Re}(f_0 * f) = \operatorname{Re}(h_0 * h + g_0 * g) = \operatorname{Re} \left( \frac{i}{2} \log \left( \frac{1-iz}{1-z} \right) + \frac{z(1-i-z)}{2(1-iz)^2} \right),$$

and

$$\operatorname{Im}(f_0 * f) = \operatorname{Im}(h_0 * h - g_0 * g) = \operatorname{Im} \left( \frac{z}{2(1-iz)} + \frac{z+z^2}{2(1-z)(1-iz)^2} \right).$$

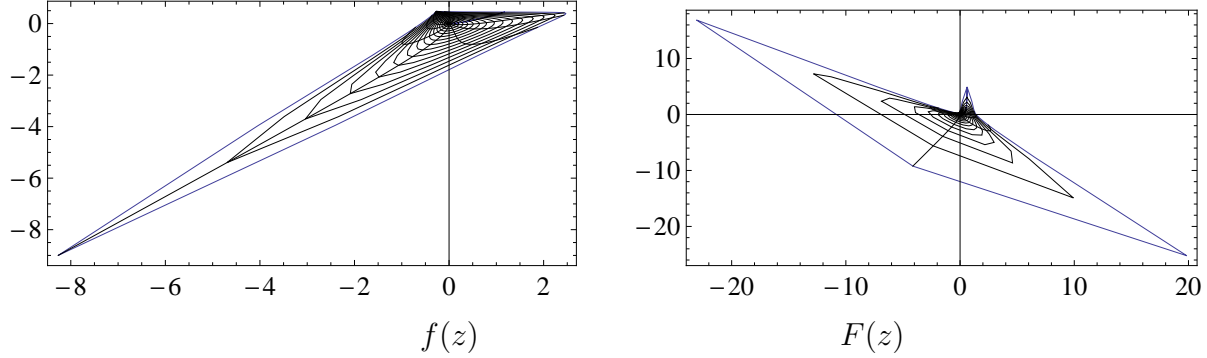
The images of the unit disk  $\mathbb{D}$  under  $f$  and  $f_0 * f$  are shown in Figure 1 as plots of the images of equally spaced radial segments and concentric circles. Similar comments apply for the other two figures in the next two examples.

If we need to exactly describe the image domains, one may proceed by introducing

$$\zeta = \frac{1-iz}{1-z} = \xi + i\eta \quad (\xi > \eta),$$

so that  $z = (\zeta - 1)/(\zeta - i)$ . Applying this transformation with  $\zeta = \xi + i\eta$ , we get

$$\operatorname{Re}(f_0 * f) = \operatorname{Re} \left( \frac{i}{2} \log \zeta + \frac{\zeta^2 - 1}{4\zeta^2} \right) = -\frac{1}{2} \arctan \frac{\eta}{\xi} + \frac{1}{4} - \frac{\xi^2 - \eta^2}{4(\xi^2 + \eta^2)^2},$$

FIGURE 1. Images of  $f$  and  $f_0 * f$ 

and

$$\begin{aligned} \operatorname{Im}(f_0 * f) &= \operatorname{Im}\left(\frac{\zeta - 1}{2(1 - i)\zeta} + \frac{(\zeta - 1)(\zeta - i)(2\zeta - 1 - i)}{2(1 - i)^3\zeta^2}\right) \\ &= \frac{1}{4} + \frac{\xi - \eta}{4} - \frac{3(\xi - \eta)}{4(\xi^2 + \eta^2)} + \frac{\xi^2 - \eta^2}{4(\xi^2 + \eta^2)^2}. \end{aligned}$$

A careful analysis may be done in order to explain the image domain of the convolution function. Here we avoid the computation although we just would like indicate the procedure for discussion.

**Example 3.** Let  $f = h + \bar{g} \in \mathcal{S}^0(H_\gamma)$  with  $\gamma = \pi$  and the dilatation  $\omega(z) = z$ . Then

$$h(z) + g(z) = \frac{z}{1 + z}, \quad h'(z) = \frac{1}{(1 + z)^3}, \quad g'(z) = \frac{z}{(1 + z)^3},$$

and therefore,

$$h(z) = \frac{z^2 + 2z}{2(1 + z)^2},$$

and

$$g(z) = \frac{z}{1 + z} - \frac{z^2 + 2z}{2(1 + z)^2} = \frac{z^2}{2(1 + z)^2}.$$

As before, we easily have  $f_0 * f = h_0 * h + \overline{g_0 * g}$  with

$$h_0 * h = \frac{h + zh'}{2} = \frac{z(z^2 + 3z + 4)}{4(1 + z)^3}$$

and

$$g_0 * g = \frac{g - zg'}{2} = -\frac{z^2(1 - z)}{4(1 + z)^3}.$$

The dilatation  $\tilde{\omega}$  of  $f_0 * f$  is

$$\tilde{\omega}(z) = z \left( \frac{z^2 + \frac{1}{2}z - \frac{1}{2}}{1 + \frac{1}{2}z - \frac{1}{2}z^2} \right).$$

Further, the real and imaginary parts of  $f_0 * f = h_0 * h + \overline{g_0 * g}$  are given by

$$\operatorname{Re}(f_0 * f) = \operatorname{Re}(h_0 * h + g_0 * g) = \operatorname{Re}\left(\frac{z(z^2 + z + 2)}{2(1 + z)^3}\right)$$

and

$$\operatorname{Im}(f_0 * f) = \operatorname{Im}(h_0 * h - g_0 * g) = \operatorname{Im} \frac{z}{(1+z)^2},$$

respectively. The images of the unit disk  $\mathbb{D}$  under  $f$  and  $f_0 * f$  are shown in Figure 2.

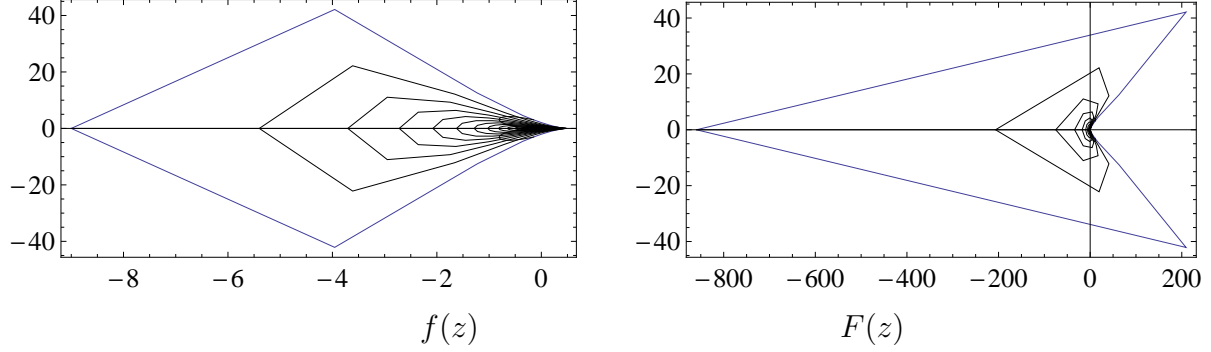


FIGURE 2. Images of  $f$  and  $f_0 * f$

If we let

$$\zeta = \frac{1-z}{1+z} = \xi + i\eta \quad (\xi > 0),$$

then  $z = (1 - \zeta)/(1 + \zeta)$  so that

$$\operatorname{Re}(f_0 * f) = \frac{1}{8} \operatorname{Re}(-\zeta^3 - \zeta + 2) = -\frac{\xi^3}{8} + \frac{3}{8}\xi\eta^2 - \frac{\xi}{8} + \frac{1}{4}$$

and

$$\operatorname{Im}(f_0 * f) = \operatorname{Im}\left(\frac{1 - \zeta^2}{4}\right) = -\frac{\xi\eta}{2}.$$

Again, these observations suffice for the discussion of the image domain of  $f_0 * f$ .

**Example 4.** Let  $f = h + \bar{g}$  be a harmonic mapping of  $\mathbb{D}$  such that

$$h(z) + g(z) = \frac{z}{1+z} \quad \text{and} \quad \omega(z) = -z^2.$$

Then  $g'(z) = -z^2 h'(z)$  and as before, we see that

$$h'(z) = \frac{1}{(1-z)(1+z)^3} \quad \text{and} \quad g'(z) = -\frac{z^2}{(1-z)(1+z)^3}.$$

Integration gives

$$h(z) = \frac{1}{8} \log\left(\frac{1+z}{1-z}\right) + \frac{1}{4} \left( \frac{z}{1+z} - \frac{1}{(1+z)^2} + 1 \right),$$

$$g(z) = -\frac{1}{8} \log\left(\frac{1+z}{1-z}\right) + \frac{1}{4} \left( \frac{3z}{1+z} + \frac{1}{(1+z)^2} - 1 \right),$$

and  $f(\mathbb{D}) = \{w : \operatorname{Re} w < 1/2\}$ . We observe that

$$f(e^{i\theta}) = \begin{cases} \frac{1}{2} + \left( \frac{\pi}{16} + \frac{1}{4} \tan\left(\frac{\theta}{2}\right) \right) & \text{if } 0 < \theta < \pi \\ \frac{1}{2} + \left( -\frac{\pi}{16} + \frac{1}{4} \tan\left(\frac{\theta}{2}\right) \right) & \text{if } \pi < \theta < 2\pi. \end{cases}$$

Next we note that  $f_0 * f = h_0 * h + \overline{g_0 * g}$  with

$$h_0 * h = \frac{1}{16} \log \left( \frac{1+z}{1-z} \right) + \frac{1}{8} \left( \frac{z}{1+z} - \frac{1}{(1+z)^2} + 1 \right) + \frac{z}{2(1-z)(1+z)^3}$$

and

$$g_0 * g = -\frac{1}{16} \log \left( \frac{1+z}{1-z} \right) + \frac{1}{8} \left( \frac{3z}{1+z} + \frac{1}{(1+z)^2} - 1 \right) + \frac{z^3}{2(1-z)(1+z)^3}.$$

It is also easy to see that  $\tilde{\omega}$  of  $f_0 * f$  is  $\tilde{\omega}(z) = z^2$ . The images of the unit disk  $\mathbb{D}$  under  $f$  and  $f_0 * f$  are shown in Figure 3.

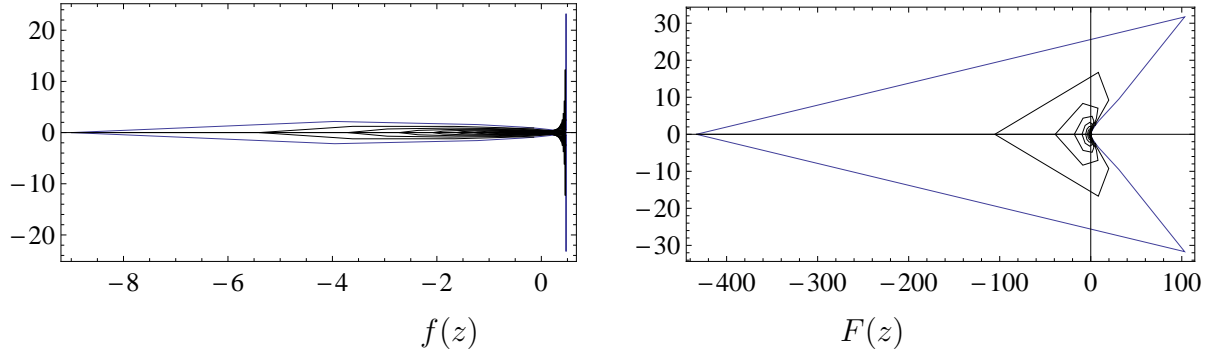


FIGURE 3. Images of  $f$  and  $f_0 * f$

Finally, if we let

$$\zeta = \frac{1+z}{1-z} = \xi + i\eta \quad (\xi > 0),$$

then  $z = (\zeta - 1)/(\zeta + 1)$  so that

$$\begin{aligned} \operatorname{Re}(f_0 * f) &= \frac{1}{2} \operatorname{Re} \left( \frac{z}{1+z} + \frac{z(1+z^2)}{2(1-z)(1+z)^3} \right) \\ &= \frac{1}{16} \operatorname{Re} \left( \zeta + 4 - \frac{4}{\zeta} - \frac{1}{\zeta^3} \right) \\ &= \frac{\xi}{16} + \frac{1}{4} - \frac{\xi}{4(\xi^2 + \eta^2)} - \frac{\xi^3 - 3\xi\eta^2}{16(\xi^2 + \eta^2)^3}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(f_0 * f) &= \operatorname{Im} \left( \frac{1}{8} \log \left( \frac{1+z}{1-z} \right) - \frac{z}{4(1+z)} - \frac{1}{4(1+z)^2} + \frac{z}{2(1+z)^2} \right) \\ &= \operatorname{Im} \left( \frac{1}{8} \log \zeta - \frac{3}{16\zeta^2} \right) \\ &= \frac{1}{8} \arctan \frac{\eta}{\xi} + \frac{3\xi\eta}{8(\xi^2 + \eta^2)^2}. \end{aligned}$$

These observations help to analyze the image of  $\mathbb{D}$  under  $f_0 * f$  and we avoid the details.

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